KAWASAKI MED. J. VOL. 10 : NO. 3, 197-206 (1984)

Estimated t Test and F Test with Interval-Censored Normal Data

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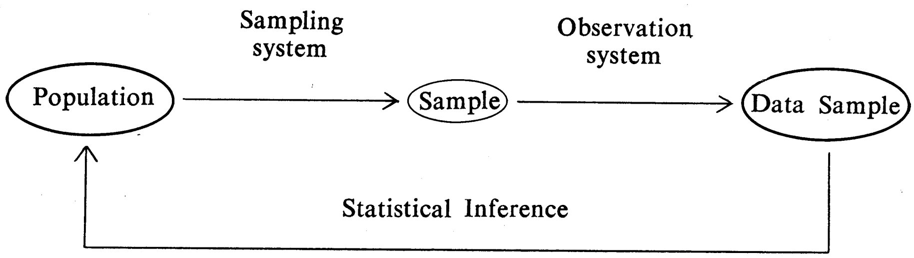
Accepted for Publication on June 28, 1984

ABSTRACT. In this paper we derive the asymptotic sampling distributions of maximum likelihood estimators based on an interval-censored data sample from a normal population and propose two test statistics analogous to the Student's t-ratio test and the Snedecor's F-ratio test in the analysis of variance. And the results of simulation studies of these tests for means are exhibited.

Key words : Interval-censored data — t test — F test

1. Introduction

It is an obvious fact that observations are dependent upon their observation systems. The same random sample produces different sets of sample values, according to different observation systems. In this sense we are practically obliged to deal with a random data sample depending on a specified observation system.



As examples of observation system we can adduce periodical inspection, periodical medical checkup and appointment system in medical examination. In these cases the random variables under studies — a life span of machine part, an age when a specified deciduous tooth erupted and a period of time from a surgical operation to relapse of the disease — are continuous but, in general, we cannot observe their realized values exactly. A continuous random variable X is said to be interval-censored into a non-zero interval I if the only information about a realized value of X is that the realized value lies in I. We call a random data sample an interval-censored data sample if all the individual values are interval-censored.

We will discuss the properties of maximum likelihood estimators based on an interval-censored data sample. In this connection, it is clear that the

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discussion below is applicable to a grouped data sample, because a grouped data sample is the special case of an interval-censored data sample.

2. Notations and definitions

Let X_α be a continuous random variable with values over (ξ, η) $(-\infty \leq \xi < \eta \leq \infty)$, and its distribution function be $F(x, \theta)$, where θ is an unknown parameter vector. The parameter space is assumed to be an open set in an m dimensional Euclidian space R^m and will be denoted by Θ.

Let $\mathbf{x}_\alpha = (x_{\alpha 0}, x_{\alpha 1}, \ldots, x_{\alpha s_\alpha}, x_{\alpha, s_\alpha+1})$ be a censoring observation vector for X_α, where $\xi \equiv x_{\alpha 0} < x_{\alpha 1} < \ldots < x_{\alpha s_\alpha} < x_{\alpha, s_\alpha+1} \equiv \eta$ and $x_{\alpha 1}$, $x_{\alpha 2}, \ldots, x_{\alpha s_\alpha}$ are a sequence of constants independent of realized value of X_α and θ.

Then we have a multinomial random variable

$$\Delta_\alpha \equiv \Delta(X_\alpha, \mathbf{x}_\alpha) = (\Delta_{\alpha 0}, \Delta_{\alpha 1}, \ldots, \Delta_{\alpha s_\alpha})$$

defined for the pair $(X_\alpha, \mathbf{x}_\alpha)$ by

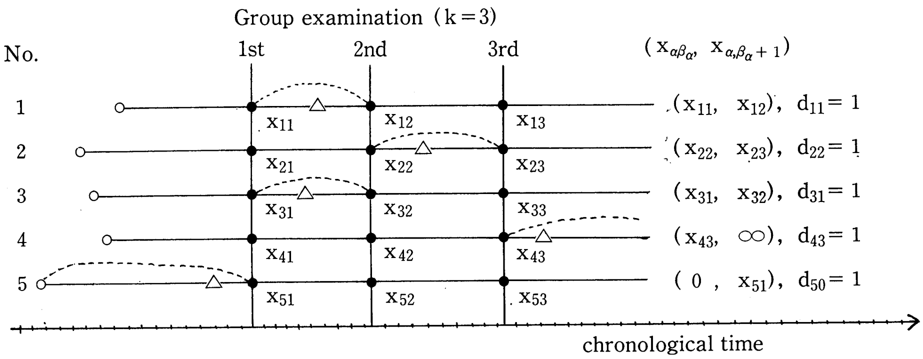
$$\Delta_{\alpha\beta} = \begin{cases} 1 & \text{if } X_\alpha \in (x_{\alpha\beta}, x_{\alpha,\beta+1}), \\ 0 & \text{if } X_\alpha \notin (x_{\alpha\beta}, x_{\alpha,\beta+1}), \quad \beta = 0, \ldots, s_\alpha. \end{cases}$$

It is clear that $\Pr\{\Delta_{\alpha\beta} = 1\} = F(x_{\alpha,\beta+1}, \theta) - F(x_{\alpha\beta}, \theta)$ and $\sum_{\beta=0}^{s_\alpha} \Pr\{\Delta_{\alpha\beta} = 1\} = 1$.

Now let X_α ; $\alpha = 1, \ldots, n$ be independently and identically distributed random variables with distribution function $F(x, \theta)$ and with censoring observation vector $\mathbf{x}_\alpha$; $\alpha = 1, \ldots, n$ individually, where $\mathbf{x}_\alpha$'s are not necessarily different. Then we have a set of n independent observations

$$\Delta_\alpha = (\Delta_{\alpha 0}, \Delta_{\alpha 1}, \ldots, \Delta_{\alpha s_\alpha}) \ ; \ \alpha = 1, \ldots, n.$$

$\{(\Delta_\alpha, \mathbf{x}_\alpha) ; \alpha = 1, \ldots, n\}$ constitutes an interval-censored data sample.



○=born, △=erupted, ●=censoring observed,

$x_{\alpha\beta}$=an age of the α-th child at the β-th group examination

Illustration of an Interval-Censored Data Sample

—An eruption age of specified deciduous tooth—

The log-likelihood function of this interval-censored data sample is given by

(1) $\log L(\theta) \propto \sum_{\alpha=1}^{n} \sum_{\beta=0}^{s_\alpha} \Delta_{\alpha\beta} \log[F(x_{\alpha,\beta+1}, \theta) - F(x_{\alpha\beta}, \theta)]$.

Let $\mathbf{d}_\alpha$ be an observed vector of Δ_α with $d_{\alpha\beta_\alpha} = 1$ for some β_α ($0 \leq \beta_\alpha \leq s_\alpha$). Then the log-likelihood of the observed interval-censored data sample is given by

(2) $\log \ell(\theta) \propto \sum_{\alpha=1}^{n} \log[F(x_{\alpha,\beta_\alpha+1}, \theta) - F(x_{\alpha\beta_\alpha}, \theta)]$.

The maximum likelihood estimate (MLE) $\hat{\theta}$ of the parameter θ is defined as the value of $\theta \in \Theta$ which attains the supremum of $\log \ell(\theta)$, i.e.,

(3) $\log \ell(\hat{\theta}) = \sup_{\theta \in \Theta} \log \ell(\theta)$.

Since the parameter space Θ is an open set, it may sometimes happen that the supremum in (3) is not attained at an interior point of the parameter space Θ, i.e., the MLE does not exist. If the MLE is indeterminable because $\log \ell(\theta)$ attains its maximum everywhere in a certain region of the parameter space, we will regard the situation as equivalent to the one where the MLE does not exist.

3. Existence of MLE in a normal population

When the distribution function $F(x, \theta)$ is a normal distribution function, the existence theorems of MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ based on an interval-censored data sample have been developed in our previous paper.[6)]

Namely, using the condition

(C) : $x_{\alpha 1} \leq x_{\alpha\beta_\alpha} \leq x_{\alpha,s_\alpha-1}$ hold for at least one $\alpha = 1, \ldots, n$

and another condition

(C*) : Putting $\zeta = x_{\alpha\beta_\alpha}$ or $x_{\alpha,\beta_\alpha+1}$ for any $\alpha = 1, \ldots, n$, $x_{\alpha',\beta_{\alpha'}+1} < \zeta$ or $\zeta < x_{\alpha'\beta_{\alpha'}}$ holds for at least one $\alpha' \in \{1, \ldots, n\}$ in either case of ζ,

the existence theorems are expressed as follows.

Theorem 1. Let $\{\mathbf{d}_\alpha ; d_{\alpha\beta_\alpha} = 1, \alpha = 1, \ldots, n\}$ be an observed interval-censored data sample of a random sample $\{X_\alpha ; \alpha = 1, \ldots, n\}$ from a normal distribution function $F(x, \theta)$ with respect to the censoring observation vector $\{\mathbf{x}_\alpha ; \alpha = 1, \ldots, n\}$. Assume that $\{\mathbf{d}_\alpha ; d_{\alpha\beta_\alpha} = 1, \alpha = 1, \ldots, n\}$ satisfies the condition (C). Then the MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ based on $\{\mathbf{d}_\alpha ; d_{\alpha\beta_\alpha} = 1, \alpha = 1, \ldots, n\}$ exists if and only if $\{\mathbf{d}_\alpha ; d_{\alpha\beta_\alpha} = 1, \alpha = 1, \ldots, n\}$ satisfies the condition (C*).

Theorem 2. Assume that $\{\mathbf{d}_\alpha ; d_{\alpha\beta_\alpha} = 1, \alpha = 1, \ldots, n\}$ does not satisfy the condition (C), but fulfills the condition (C*). Let A be the set of α with $d_{\alpha 0} = 1$, and let $x_{\alpha\cdot}$ be

$$x_{\alpha\cdot} = \begin{cases} x_{\alpha s_\alpha} & \text{if } \beta_\alpha = s_\alpha, \\ x_{\alpha 1} & \text{if } \beta_\alpha = 0. \end{cases}$$

If $0 < n(A) < n$ hold and the inequality

$$\frac{\sum_{\alpha=1}^{n} x_{\alpha\cdot}}{n} < \frac{\sum_{A} x_{\alpha 1}}{n(A)} \tag{4}$$

holds, then the MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ based on $\{\mathbf{d}_\alpha ; d_{\alpha\beta_\alpha} = 1, \alpha = 1, \ldots, n\}$ exists, where $\sum_{A}$ implies the summation over the set A and n(A) is the cardinality of A.

The theorem 1 is mainly concerned with a sample whose variable has a censoring observation system $\mathbf{x}_\alpha$ with $s_\alpha \geq 2$ for almost all α, and the theorem 2 is mostly connected with a sample whose variable has a censoring observation system $\mathbf{x}_\alpha$ with $s_\alpha = 1$ for almost all α.

4. The asymptotic variance-covariance matrix of MLE's and their asymptotic efficiencies

Interval-censoring causes a loss of information but has the merit of easier collection of data. We can measure our loss of information by using the desirable properties of MLE.

For any sample size n, there is a positive probability that the MLE $\hat{\theta}$ for an interval-censored data sample will not exist. Therefore, the MLE $\hat{\theta}$ cannot be considered as a random variable in the strict sense. Since the asymptotic properties discussed below have reference to an infinite sequence of random variables, we consider the random variable which has the distribution function $\Pr\{\hat{\theta} \leq x \mid \hat{\theta} \text{ exists}\}$ instead of $\hat{\theta}$. This random variable will be called the conditional MLE of θ and be denoted by $\hat{\hat{\theta}}$.

Let X_α ; $\alpha = 1, 2, \ldots, n$ be independently and identically distributed random variables each of which has the same normal distribution function $F(x ; \mu, \sigma^2)$ and an individual censoring observation vector $\mathbf{x}_\alpha$; $\alpha = 1, 2, \ldots,$ n respectively, where $\mathbf{x}_\alpha$'s are not necessarily different. When $n \to \infty$, the censoring observation vector $\mathbf{x}_\alpha$ for X_α may be assumed to be selected at random out of m different censoring observation systems.

Assuming that the both parameters μ and σ^2 are unknown, we consider the maximum likelihood simultaneous estimate (MLSE) $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ based on an interval-censored data sample. In this case, MLSE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ is obtained as the solution of normal equations by the improved Newton-Raphson iterative method uniquely under the theorem 1 or 2. It is known that MLSE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ is consistent and asymptotically efficient if it exists, and $\Pr\{\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ does not exist$\}$ will tend to zero as $n \to \infty$. Therefore, the conditional MLSE $\hat{\hat{\theta}} = (\hat{\hat{\mu}}, \hat{\hat{\sigma}}^2)$ with the distribution function $\Pr\{\hat{\theta} \leq x \mid \hat{\theta} \text{ exists}\}$ will be consistent and asymptotically efficient. Then the following relation for the variance-covariance matrix of $\hat{\hat{\mu}}$ and $\hat{\hat{\sigma}}^2$ will hold for sufficiently large n :

$$(5)\quad n\cdot V(\hat{\mu},\hat{\sigma}^2) \doteq \begin{bmatrix} \frac{1}{\sigma^2 n}\sum\sum\frac{(\varDelta\phi_{\alpha\beta})^2}{\varDelta\Phi_{\alpha\beta}} & \frac{1}{2\sigma^3 n}\sum\sum\frac{(\varDelta\phi_{\alpha\beta})(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})}{\varDelta\Phi_{\alpha\beta}} \\ \frac{1}{2\sigma^3 n}\sum\sum\frac{(\varDelta\phi_{\alpha\beta})(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})}{\varDelta\Phi_{\alpha\beta}} & \frac{1}{4\sigma^4 n}\sum\sum\frac{(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})^2}{\varDelta\Phi_{\alpha\beta}} \end{bmatrix}^{-1},$$

where every $\Sigma\Sigma$ means $\sum_{\alpha=1}^{n}\sum_{\beta=0}^{s_\alpha}$ and $z_{\alpha\beta} = (x_{\alpha\beta}-\mu)/\sigma$, $\varDelta\phi_{\alpha\beta} = \phi(z_{\alpha,\beta+1}) - \phi(z_{\alpha\beta})\dagger$, $\varDelta z_{\alpha\beta}\phi_{\alpha\beta} = z_{\alpha,\beta+1}\phi(z_{\alpha,\beta+1}) - z_{\alpha\beta}\phi(z_{\alpha\beta})$, $\varDelta\Phi_{\alpha\beta} = \Phi(z_{\alpha,\beta+1}) - \Phi(z_{\alpha\beta})\dagger$.

On the other hand, it is a well-known fact that for an exactly observed data sample we have

$$(6)\quad \lim_{n\to\infty} n\cdot V(\hat{\mu},\hat{\sigma}^2) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}^{-1}.$$

The right hand of (5) will tend to (6) as $n\to\infty$, when every censoring observation points $(x_{\alpha 0}, x_{\alpha 1}, \cdots, x_{\alpha s_\alpha}, x_{\alpha, s_\alpha+1})$ grows infinitely fine.

From (5) and (6) we can derive the asymptotic efficiency of $\hat{\mu}$ and the one of $\hat{\sigma}^2$ as follows ~~respectively~~ :

$$(7)\quad e(\hat{\mu}) \doteq \frac{1}{n}\sum\sum\frac{(\varDelta\phi_{\alpha\beta})^2}{\varDelta\Phi_{\alpha\beta}} - \frac{1}{2}\left(\frac{1}{n}\sum\sum\frac{(\varDelta\phi_{\alpha\beta})(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})}{\varDelta\Phi_{\alpha\beta}}\right)^2 \Big/ \frac{1}{2n}\cdot\sum\sum\frac{(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})^2}{\varDelta\Phi_{\alpha\beta}},$$

$$(8)\quad e(\hat{\sigma}^2) \doteq \frac{1}{2n}\sum\sum\frac{(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})^2}{\varDelta\Phi_{\alpha\beta}} - \frac{1}{2}\left(\frac{1}{n}\sum\sum\frac{(\varDelta\phi_{\alpha\beta})(\varDelta z_{\alpha\beta}\phi_{\alpha\beta})}{\varDelta\Phi_{\alpha\beta}}\right)^2 \Big/ \frac{1}{n}\cdot\sum\sum\frac{(\varDelta\phi_{\alpha\beta})^2}{\varDelta\Phi_{\alpha\beta}},$$

where every $\Sigma\Sigma$ means $\sum_{\alpha=1}^{n}\sum_{\beta=0}^{s_\alpha}$.

5. The asymptotic properties of some functions of maximum likelihood estimators

As the conditional MLSE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ will be consistent and asymptotically efficient and both $\hat{\mu}$ and $\hat{\sigma}^2$ be asymptotically normally distributed for large n respectively, the following propositions are analogized using (7) and (8). In the propositions below, $n^* = n\cdot e(\hat{\mu})$ and $n^{**} = n\cdot e(\hat{\sigma}^2)$.

(9) The random variable $\hat{\mu}$ has an asymptotic normal distribution $N(\mu, \sigma^2/n^*)$.

(10) The random variable $n^{**}\hat{\sigma}^2$ has an asymptotic χ^2 distribution with $n^{**} - 1$ degrees of freedom.

(11) The random variable $\hat{t} = \sqrt{n^*}(\hat{\mu}-\mu)/\sqrt{n^{**}\hat{\sigma}^2/(n^{**}-1)}$ has an

†) $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and distribution function of the standard normal distribution respectively.

asymptotic t distribution with $n^{**}-1$ degrees of freedom.

(12) For two samples, the random variable

$$\hat{t}=\frac{(\hat{\mu}_1-\hat{\mu}_2)-(\mu_1-\mu_2)}{\sqrt{1/n_1^{*}+1/n_2^{*}}\ \sqrt{(n_1^{**}\hat{\sigma}_1^2+n_2^{**}\hat{\sigma}_2^2)/(n_1^{**}+n_2^{**}-2)}}$$

has an asymptotic t distribution with $n_1^{**}+n_2^{**}-2$ degrees of freedom, where suffix i corresponds to each normal population with common variance respectively.

(13) For r (≥ 3) samples, the random variable

$$\hat{F}=\frac{\sum_1^r n_i^{*}(\hat{\mu}_i-\hat{\mu}_T)^2/(r-1)}{\sum_1^r n_i^{**}\hat{\sigma}_i^2 \Big/ \sum_1^r (n_i^{**}-1)}$$

has an asymptotic F distribution with $(r-1, \sum_1^r(n_i^{**}-1))$ degrees of freedom under the null hypothesis $H:\mu_1=\mu_2=\ldots=\mu_r$ and $\sigma_1^2=\sigma_2^2=\ldots=\sigma_r^2$, where $\hat{\mu}_T$ is the MLE of mean of overall population.

If the sample size n_i is not so small and censoring observation vector $\mathbf{x}_i$ is not inadequate extremely, we can perform the statistical inference approximately by means of (9) $\sim$ (13).

6. Results of simulation studies

In the previous paragraph, the distributions of the maximum likelihood estimators were analogized only for sufficiently large n of observations or n $\rightarrow\infty$. Here the distributions of estimated t-ratio, sum of squares and F-ratio will be investigated for n limited.

In order not to make things unduly complicated, we consider the following simulation model :

$$X_{ij}=\alpha+\beta_i+\varepsilon_{ij},\quad \begin{array}{l} i=1,\ldots,r,\\ j=1,\ldots,n_i,\end{array}\quad \sum_1^r\beta_i=0,\ \sum_1^r n_i=n,$$

where ε_{ij}'s are independently and normally distributed with zero mean and variance σ^2. Furthermore, let the censoring observation vector be all common, namely, $\mathbf{x}_{ij}=\mathbf{x}=(-\infty, x_1,\ldots, x_k, \infty)$ $(k\geq 2)$.

In this model, an interval-censored data sample degenerates into a grouped data sample, and the estimated quantities will increase in discreteness. Accordingly, the sampling distribution of the quantity will be worse accordance with the asymptotic distribution.

Now we consider the MLSE $\hat{\theta}=(\hat{\mu}, \hat{\sigma}^2)$. In this case MLSE's $(\hat{\mu}, \hat{\sigma}^2)$ based on interval-censored data samples are obtained as the solutions of normal equations by the improved Newton-Raphson iterative method uniquely under the condition that all values do not fall into neighbouring any two intervals

(including both end intervals). Practically we put
$r = 4$, $n_1 = n_2 = n_3 = n_4 = 20$, $n = 80$,
$\alpha = 0$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (-0.5, 0.0, 0.2, 0.3)$, $\sigma = 1$, and
$k = 8$, $(x_1, x_2, \ldots, x_8) = (-3.5, -2.5, -1.5, -0.5, 0.5, 1.5, 2.5, 3.5)$.
Then we have

$$X_{1j} \sim N(-0.5, 1),$$
$$X_{2j} \sim N(0, 1),$$
$$X_{3j} \sim N(0.2, 1),$$
$$X_{4j} \sim N(0.3, 1).$$

From (7) and (8), we get

$$e(\hat{\mu}) \doteq 0.923 \text{ and } e(\hat{\sigma}^2) \doteq 0.854$$

in every X_{ij} and their aggregation, and so $n_i^* \doteq 18.46$, $n_i^{**} \doteq 17.08$ and $n^{**} \doteq 68.32$.

We repeated the generation of four standard normal samples (size 20) 1000 times and obtained the following results.

In the following tables, t_i, SS_i and F_i are ordinary t-ratio, sum of squares and F-ratio based on the exactly observed data sample respectively. And the figures in * column (or row) show the frequency of values being significant at 5% level but not at 1% level, and the figures in ** column (or row) show the frequency of values being significant at 1% level.

TABLE 1. Correlation table of t and $\hat{t}$.

$\hat{t}_1$			*	**	Total	
t_1		936	16	0	952	
	*	11	21	8	40	48
	**	0	3	5	8	
	Total	947	40	13	1000	
			53			

$\hat{t}_2$			*	**	Total	
t_2		933	16	2	951	
	*	16	22	4	42	49
	**	0	0	7	7	
	Total	949	38	13	1000	
			51			

$\hat{t}_3$			*	**	Total	
t_3		926	14	1	941	
	*	19	26	3	48	59
	**	0	3	8	11	
	Total	945	43	12	1000	
			55			

$\hat{t}_4$			*	**	Total	
t_4		922	13	0	935	
	*	13	24	10	47	65
	**	2	6	10	18	
	Total	937	43	20	1000	
			63			

$$\hat{t}_i = \frac{\hat{\mu}_i-\mu_i}{\hat{\sigma}_i}\sqrt{n_i^*}\sqrt{\frac{n_i^{**}-1}{n_i^{**}}} \;\dot{\sim}\; t_{n_i^{**}-1}, \qquad (\text{cf. } t_i \sim t_{19})$$

$n_i^* \doteq 18.46, \quad n_i^{**} \doteq 17.08$

TABLE 2. Correlation table of SS_i and $\widehat{\widehat{SS}}_i$.

$\widehat{\widehat{SS}}_1$			*	**	Total
SS_1		946	15	0	961
	*	14	19	2	35 } 39
	**	0	2	2	4 }
	Total	960	36	4	1000
			$\underbrace{36 + 4}_{40}$		

$\widehat{\widehat{SS}}_2$			*	**	Total
SS_2		940	12	0	952
	*	17	20	4	41 } 48
	**	0	4	3	7 }
	Total	957	36	7	1000
			$\underbrace{36 + 7}_{43}$		

$\widehat{\widehat{SS}}_3$			*	**	Total
SS_3		947	10	0	957
	*	14	19	2	35 } 43
	**	0	1	7	8 }
	Total	961	30	9	1000
			$\underbrace{30 + 9}_{39}$		

$\widehat{\widehat{SS}}_4$			*	**	Total
SS_4		929	19	1	949
	*	13	23	5	41 } 51
	**	1	3	6	10 }
	Total	943	45	12	1000
			$\underbrace{45 + 12}_{57}$		

$\widehat{\widehat{SS}}_i = n_i^{**}\widehat{\sigma}_i^2 \ \dot{\sim}\ \chi^2_{n_i^{**}-1}$, (cf. $SS_i \sim \chi^2_{19}$)

$n_i^{**} \doteq 17.08$

TABLE 3. Correlation table of F and $\hat{\hat{F}}$.

	$\hat{\hat{F}}$		*	**	Total
F		351	44	0	395
	*	40	139	35	214 } 605
	**	0	37	354	391 }
	Total	391	220	389	1000
			609		

$$\hat{\hat{F}} = \frac{\sum_{1}^{r} n_i^* (\hat{\mu}_i - \hat{\mu}_T)^2 / (r-1)}{\sum_{1}^{r} n_i^{**} \hat{\sigma}_i^2 \Big/ \sum_{1}^{r} (n_i^{**}-1)} \; \dot{\sim} \; F^{\,r-1}_{\,\Sigma(n_i^{**}-1)}\,, \qquad (\text{cf. } F \sim F^3_{76})$$

$n_i^* \doteq 18.46, \quad n_i^{**} \doteq 17.08$

These correlation tables show that estimated t-ratio based on grouped data (degenerated interval-censored data), say $\hat{t}_i$, are good accordance with t_i-ratio based on exactly observed data from a viewpoint of testing, and a similar result is obtained in the relation between $\hat{\hat{F}}$-ratio and F-ratio.

Therefore, we can perform the statistical tests approximately by means of $\hat{t}$ and $\hat{\hat{F}}$, if a sample size is not so small and a censoring observation vector is not so inadequate.

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