Existence Theorems for an Optimal Solution in the Weighted Least Squares Problem

by

Tadashi NAKAMURA

Department of Mathematics, Kawasaki Medical School
Kurashiki 701-01, Japan
(Received on Oct. 3, 1978)

Introduction

In [2], a weighted least squares problem of finding the vector θ in the parameter space Θ which minimizes $\sum_{i=1}^{m}[F(x_i, \theta)(1-F(x_i, \theta))]^{-1}(F(x_i, \theta)-y_i)^2$ over Θ is discussed and the existence theorem is proved. In the present paper, we consider a general weighted least squares problem involving that of [2] as a special case. The method used in [2] to prove the existence theorem can be also applicable to our case and a sufficient condition for the existence of an optimal solution is given in § 3. In § 4, the existence theorem (Theorem 2) is applied to a least squares problem with specified weight function.

§ 1. Preliminaries and notations

Let Θ be a domain of the p-dimensional Euclidean space R^p. Denote by $d(\theta, \Theta')$ the distance between θ and a nonempty subset Θ' of R^p. We regard $\{\infty\}$ as a boundary point of Θ if Θ is unbounded and define $d(\theta, \{\infty\})\equiv d(\theta, \{0\})^{-1}$. For each $\theta\in\Theta$, let $F(x, \theta)$ be a real valued function defined on the interval $I=(a, b)$ $(-\infty\leqq a<b\leqq\infty)$ in the real line R such that I is independent of θ, $0<F(x, \theta)<1$, $\lim_{x\to a} F(x, \theta)=0$ and $\lim_{x\to b} F(x, \theta)=1$. We always assume that $\{x_i\}$ and $\{y_i\}$ are sets of m real numbers such that $a<x_1<x_2<\cdots<x_m<b$ and $0\leqq y_1\leqq y_2\leqq\cdots\leqq y_m\leqq 1$. For each $u\in(0, 1)$ and $\theta\in\Theta$, we define functions $\overline{Q}(u)$ and $Q(\theta)$ by

$$\overline{Q}(u)\equiv\sum_{i=1}^{m} w(u)(u-y_i)^2 \text{ and } Q(\theta)\equiv\sum_{i=1}^{m} w(F(x_i, \theta))(F(x_i, \theta)-y_i)^2,$$

where $w(u)$ is a positive differentiable function defined on $(0, 1)$ such that:

(1. 1) $\lim_{u\to 0} w(u)=\lim_{u\to 1} w(u)=\infty$,

(1. 2) $\overline{\lim_{u\to 0}}\, u^2w(u)$ and $\overline{\lim_{u\to 1}}\,(1-u)^2w(u)$ are finite.

We consider the following extremum problem which is called the least squares problem:

(P) Choose $\theta \in \Theta$ so that θ minimizes $Q(\theta)$.

Our main purpose is to prove the existence of an optimal solution of the problem (P). Throughout this paper we always assume that $F(x, \theta)$ can be written as

$$F(x, \theta)=\int_{-\infty}^{t(x,\ \theta)} f(v)dv,$$

where $f(v)$ is a positive integrable function on R and $t(x, \theta)$ is a function on $I\times\Theta$ satisfying the following conditions:

(1. 3) $t(x, \theta)$ is continuous in θ for fixed x and is strictly increasing in x for fixed θ,

(1. 4) $\{t(x, \theta)\,;\ \theta\in\Theta\}=(-\infty, \infty)$ for each $x\in I$,

(1. 5) The boundary $\partial\Theta$ of Θ is decomposed into two disjoint nonempty subsets Θ_1 and Θ_2 such that for $x, x'\in I$ with $x\neq x'$, $\lim_{\varepsilon\to+0}\inf\{|t(x, \theta)-t(x', \theta)|\,;\ \theta\in\Theta$ and $d(\theta, \Theta_1)\leqq\varepsilon\}=\infty$ and $\lim_{\varepsilon\to+0}\sup\{|t(x, \theta)-t(x', \theta)|\,;\ \theta\in\Theta$ and $d(\theta, \Theta_2)\leqq\}\varepsilon=0$.

§ 2. The representation of the s-boundary of $F(\Theta)$

Let S be a nonempty subset of R^m. We define a subset $\partial_s S$ of R^m which is called the s-boundary of S as follows: A point $z\in R^m$ belongs to $\partial_s S$ if and only if $z\notin S$ and there exists a sequence $\{z_k\}$ in S which converges to z. It is clear that $\partial_s S=\bar{S}\backslash S$, where $\bar{S}$ is the closure of S. Let $F(\theta)$ denote the row vector with components $F(x_i, \theta)$ and define the line segment L_i $(1\leqq i\leqq m)$ as $\{z=(z_1, z_2, \cdots, z_m)\in R^m\,;\ z_j=0$ $(j<i)$, $0\leqq z_i\leqq 1$ and $z_j=1$ $(i<j)\}$ and the line segment L_o as $\{z=(z_1, z_2, \cdots, z_m)\in R^m\,;$ $z_1=z_2=\cdots=z_m$ and $0\leqq z_1\leqq 1\}$. For each $x\in I$ and $v\in R$, we define a subset $S(x, v)$ of Θ by $S(x, v)\equiv\{\theta\in\Theta\,;\ t(x, \theta)=v\}$. Hereafter we always assume that

(2. 1) $\partial_s S(x_i, v)\cap\Theta_1\neq\phi$ and $\partial_s S(x_i, v)\cap\Theta_2\neq\phi$ for each i and v.

Note that the image $F(\Theta)$ of Θ under F is a bounded subset of R^m. Now we show that $\partial_s F(\Theta)$ can be represented by $L_i (0\leqq i\leqq m)$.

THEOREM 1. $\partial_s F(\Theta)=\bigcup_{i=0}^{m} L_i$.

PROOF. Put $L=\bigcup_{i=o}^{m} L_i$. It should be noted that $L\cap F(\Theta)=\phi$. At first we shall show $L\subset\partial_s F(\Theta)$. Let $z=(z_1, z_2, \cdots, z_m)\in L_i (1\leqq i\leqq m)$. Then $z_j=0$ $(j<i)$, $0\leqq z_i\leqq 1$ and $z_j=1$ $(i<j)$. In case $0<z_i<1$, we can see by (1. 4) that there exists $\theta_o\in\Theta$ such as $F(x_i, \theta_o)=z_i$. Since $\partial_s S(x_i, t(x_i, \theta_o))\cap\Theta_1\neq\phi$, there is a sequence $\{\theta_n\}$ in $S(x_i, t(x_i, \theta_o))$ such that $\lim_{n\to\infty} d(\theta_n, \Theta_1)=0$. For any $\varepsilon>0$ we have

$$|t(x_j, \theta_n)-t(x_i, \theta_n)|\geqq \inf\{|t(x_j, \theta)-t(x_i, \theta)|;\ \theta\in\Theta \text{ and } d(\theta, \Theta_1)\leqq\varepsilon\}$$

for all n with $d(\theta_n, \Theta_1)\leqq\varepsilon$ and $i\neq j$. Since $t(x_i, \theta_n)$ is constant, (1. 5) yields that $\{|t(x_j, \theta_n)|\}$ converges to ∞, and consequently we have $\lim_{n\to\infty} t(x_j, \theta_n)=-\infty (j<i)$ and $\lim_{n\to\infty} t(x_j, \theta_n)=\infty$ $(i<j)$ by (1. 3). This shows that $F(\theta_n)\to z$ as $n\to\infty$, so that $z\in\partial_s F(\Theta)$. In case $z_i=0$, we can find a sequence $\{z_n\}$ in $L_i\backslash(L_{i-1}\cup L_{i+1})$ such that $z_n\to z$ as $n\to\infty$ and a sequence $\{\theta_{nk}\}$ in Θ such that $F(\theta_{nk})\to z_n$ as $k\to\infty$ by the same reasoning as in the case treated above. Take an integer k_n so that $|F(\theta_{nk_n})-z_n|<1/n$. Then it is easily seen that $F(\theta_{nk_n})\to z$ as $n\to\infty$, so that $z\in\partial_s F(\Theta)$. By the same way we have $z\in\partial_s F(\Theta)$ for the case $z_i=1$. Finally we consider the case $z=(z_1, z_2, \cdots, z_m)\in L_o$. We have already shown $z\in\partial_s F(\Theta)$ for the case that $z_1=0$ or 1, so we may assume that $0<z_1<1$. Then by (1. 4) we can find $\theta_o\in\Theta$ such as $F(x_1, \theta_o)=z_1$. Since $\partial_s S(x_1, t(x_1, \theta_o))\cap\Theta_2\neq\phi$, there is a sequence $\{\theta_n\}$ in $S(x_1, t(x_1, \theta_o))$ such that $\lim_{n\to\infty} d(\theta_n, \Theta_2)=0$. For any $\varepsilon>0$ we have

$$|t(x_j, \theta_n)-t(x_1, \theta_n)|\leqq \sup\{|t(x_j, \theta)-t(x_1, \theta)|;\ \theta\in\Theta \text{ and } d(\theta, \Theta_2)\leqq\varepsilon\}$$

for all n with $d(\theta_n, \Theta_2)\leqq\varepsilon$ and $j>1$. (1. 5) implies that $t(x_j, \theta_n)\to t(x_1, \theta_o)$ as $n\to\infty$ for all j, so that $F(\theta_n)\to z$ as $n\to\infty$ and $z\in\partial_s F(\Theta)$. Therefore $L\subset\partial_s F(\Theta)$.

Next we shall establish the converse inclusion. Let $z\in\partial_s F(\Theta)$. Then there is a sequence $\{\theta_n\}$ in Θ such that $F(\theta_n)\to z$ as $n\to\infty$ and $z\notin F(\Theta)$. This implies that the sequence $\{\theta_n\}$ has no cluster point in Θ. Hence we may assume that $\lim_{n\to\infty} d(\theta_n, \Theta_1)=0$ or $\lim_{n\to\infty} d(\theta_n, \Theta_2)=0$. Consider m sequences $\{t(x_i, \theta_n)\}$ $(i=1, \cdots, m)$, and suppose firstly that one of these sequences has a convergent subsequence with a finite limit. Without loss of generality we may assume that $\{t(x_i, \theta_n)\}$ converges to a finite limit. In case that $\lim_{n\to\infty} d(\theta_n, \Theta_2)=0$, $\lim_{n\to\infty} t(x_j, \theta_n)=\lim_{n\to\infty} t(x_i, \theta_n)$ for all j by (1. 5). Thus $z\in L_o\subset L$. In case that $\lim_{n\to\infty} d(\theta_n, \Theta_1)=0$, $\lim_{n\to\infty} t(x_j, \theta_n)=-\infty (j<i)$ and $\lim_{n\to\infty} t(x_j, \theta_n)=\infty (i<j)$ by (1. 5), so that $z\in L_i\subset L$. Secondly we assume that each sequence $\{t(x_i, \theta_n)\}$ $(i=1, \cdots, m)$ has no convergent subsequence with a finite limit; so we may assume that for each i, $\{t(x_i, \theta_n)\}$ converges to $-\infty$ or ∞. The monotony of $t(x, \theta)$ in x yields that $z\in\bigcup_{i=1}^{m} L_i\subset L$. Thus $\partial_s F(\Theta)\subset L$. This completes the proof.

§ 3. Existence theorem

In this section we shall give a sufficient condition for which an optimal solution of the problem (P) exists. Suppose that there is $\bar{x}\in I$ such that for each $v\in R$, there are mappings $\theta(r)$ and $q(r)$ defined on $(0, \infty)$ such that:

(3. 1) $\{\theta(r);\ r\in(0, \infty)\}\subset S(\bar{x}, v)$ and $d(\theta(r), \Theta_2)\to 0$ as $r\to\infty$,

(3. 2) $t(x, \theta(r))$ is differentiable on $(0, \infty)$ for each $x \in I$,

(3. 3) $q(r)$ is positive on $(0, \infty)$, $\lim_{r\to\infty} q(r)dt(x, \theta(r))/dr \equiv T(x)$ exists and $T(x)$ is decreasing in x.

Let $u \in (0, 1)$ and let $F^{-1}(u)$ be the inverse function of the function $u = \int_{-\infty}^{t} f(s)ds$. We set

$$g(u) \equiv 2f(F^{-1}(u))w(u)\{\sum_{i=1}^{m}(u-y_i)^2\}^{-1},\ G(u) \equiv (m\sum_{i=1}^{m} y_i T(x_i) - \sum_{i=1}^{m} y_i \sum_{i=1}^{m} T(x_i))u^2 - (m\sum_{i=1}^{m} y_i^2 T(x_i)$$
$$- \sum_{i=1}^{m} y_i^2 \sum_{i=1}^{m} T(x_i))u + \sum_{i=1}^{m} y_i \sum_{i=1}^{m} y_i^2 T(x_i) - \sum_{i=1}^{m} y_i^2 \sum_{i=1}^{m} y_i T(x_i).$$

For each $k(0\leqq k\leqq m)$, denote by $\mathring{L}_k$ the set excluding its terminal points from L_k. Then we have

LEMMA 1. *For any sequence* $\{\theta_n\}$ *in* Θ *with* $F(\theta_n) \to z \in \mathring{L}_k$ $(1\leqq k\leqq m)$ *as* $n\to\infty$, $\{Q(\theta_n)\}$ *is an unbounded sequence if and only if the following condition* (C_k) *is fulfilled*:

(C_k) $y_i>0$ *for some* i *with* $i<k$ *or* $y_i<1$ *for some* i *with* $k<i$.

PROOF. Let $\{\theta_n\}$ be a sequence in Θ with $F(\theta_n)\to z = (z_1, \cdots, z_m) \in \mathring{L}_k$ as $n\to\infty$. Then $z_j = 0\,(j<k)$, $0<z_k<1$ and $z_j = 1\,(k<j)$. Put $a(\theta_n) = \sum_{i=1}^{k-1} w(F(x_i, \theta_n))(F(x_i, \theta_n) -y_i)^2$ and $b(\theta_n) = \sum_{i=k+1}^{m} w(F(x_i, \theta_n))(F(x_i, \theta_n)-y_i)^2$. Assume that condition (C_k) is fulfilled. If $y_i>0$ for some i with $i<k$, it then follows from (1. 1) that $w(F(x_i, \theta_n))(F(x_i, \theta_n) - y_i)^2\to\infty$ as $n\to\infty$ and hence $a(\theta_n)\to\infty$ as $n\to\infty$. Similarly we have that $b(\theta_n)\to\infty$ as $n\to\infty$ if $y_i<1$ for some i with $k<i$. Since $a(\theta_n)+b(\theta_n)\leqq Q(\theta_n)$, $\{Q(\theta_n)\}$ is an unbounded sequence. Next assume that $\{Q(\theta_n)\}$ is an unbounded sequence. Then either $\{a(\theta_n)\}$ or $\{b(\theta_n)\}$ is unbounded. Assume that condition (C_k) is not fulfilled; $y_i = 0$ $(j<k)$ and $y_i = 1\,(k<j)$. Then $a(\theta_n) = \sum_{i=1}^{k-1} w(F(x_1, \theta_n))F(x_i, \theta_n)^2$ and $b(\theta_n) = \sum_{i=k+1}^{m} w(F(x_i, \theta_n))(F(x_i, \theta_n) - 1)^2$ so that

$$\overline{\lim_{n\to\infty}}\, a(\theta_n) \leqq (k-1)\overline{\lim_{u\to 0}}\, u^2 w(u) < \infty,$$

$$\overline{\lim_{n\to\infty}}\, b(\theta_n) \leqq (m-k)\overline{\lim_{u\to 1}}\, (1-u)^2 w(u) < \infty,$$

which imply that $\{a(\theta_n)\}$ and $\{b(\theta_n)\}$ are bounded. This is a contradiction. Hence condition (C_k) is fulfilled. This completes the proof.

LEMMA 2. *Assume that there exist* j *and* j' *such that* $0<y_j<y_{j'}<1$ *and let* $\bar{u} \in (0, 1)$ *be a solution of* $d\bar{Q}(u)/du=0$. *Then there are mappings* $\theta(r)$ *and* $q(r)$ *defined on* $(0, \infty)$ *such that* (3. 1), (3. 2) *and* (3. 3) *are fulfilled replacing* v *by* $F^{-1}(\bar{u})$ *and that* $\lim_{r\to\infty} q(r)dQ(\theta(r))/dr$ $= g(\bar{u})G(\bar{u})$.

PROOF. Put $v=F^{-1}(\bar{u})$. Then there are mappings $\theta(r)$ and $q(r)$ satisfying (3. 1), (3. 2) and (3. 3). We have by (3. 2)

$$dQ(\theta(r))/dr = \sum_{i=1}^{m} \{w'(F(x_i, \theta(r)))(F(x_i, \theta(r)) - y_i)^2 + 2w(F(x_i, \theta(r)))(F(x_i, \theta(r)) - y_i)\} f(t(x_i, \theta(r)) dt(x_i, \theta(r))/dr \text{ (a. e.)},$$

where $w'(u) = dw(u)/du$. This and (1. 5) yield

$$\lim_{r\to\infty} q(r)dQ(\theta(r))/dr = \sum_{i=1}^{m} \{w'(\bar{u})(\bar{u} - y_i)^2 + 2w(\bar{u})(\bar{u} - y_i)\} f(F^{-1}(\bar{u}))T(x_i) \tag{3. 4}$$

$$= w(\bar{u})\sum_{i=1}^{m} \{w'(\bar{u})(\bar{u} - y_i)^2/w(\bar{u}) + 2(\bar{u} - y_i)\} f(F^{-1}(\bar{u}))T(x_i).$$

On the other hand, $\bar{u}$ is a solution of $d\bar{Q}(u)/du = 0$ and $w(\bar{u}) \neq 0$, so that $w'(\bar{u})/w(\bar{u}) = -2\sum_{i=1}^{m}(\bar{u} - y_i) / \sum_{i=1}^{m}(\bar{u}-y_i)^2$. Substituting this into (3. 4), we see that $\lim_{r\to\infty} q(r)dQ(\theta(r))/dr = g(\bar{u})G(\bar{u})$. Thus $\theta(r)$ and $q(r)$ are required one. This completes the proof.

THEOREM 2. *Assume that there exist j and j' such that $0 < y_j < y_{j'} < 1$ and put $U \equiv \{u' \in (0, 1)$; u' minimizes $\bar{Q}(u)$ over $(0, 1)\}$. If $G(u) > 0$ for some $u \in U$, then the problem (P) has an optimal solution.*

PROOF. It is easily verified that $U \neq \phi$ and each element of U is a solution of $d\bar{Q}(u)/du = 0$. Assume that there is $\bar{u} \in U$ such that $G(\bar{u}) > 0$. By Lemma 2 there are mappings $\theta(r)$ and $q(r)$ defined on $(0, \infty)$ such that $\{\theta(r);\ r \in (0, \infty)\} \subset \Theta$ and $\lim_{r\to\infty} q(r)dQ(\theta(r))/dr > 0$, so that there is $\theta_o \in \Theta$ such as $Q(\theta_o) < \lim_{r\to\infty} Q(\theta(r)) = \bar{Q}(\bar{u})$. We put $K = \{\theta \in \Theta;\ Q(\theta) \leqq Q(\theta_o)\}$. To prove our statement it suffices to show that K is compact. Let $\{\theta_n\}$ be any sequence in K and suppose that $\{\theta_n\}$ has no cluster point in K. Then $\{\theta_n\}$ has no cluster point in Θ since K is closed, and we can find a subsequence $\{\theta_{n'}\}$ of $\{\theta_n\}$ such that $F(\theta_{n'}) \to z \in \partial_s F(\Theta)$ as $n' \to \infty$ by the same argument as in the latter proof of Theorem 1; so we may assume that $F(\theta_n) \to z \in \partial_s F(\Theta)$ as $n \to \infty$. Note that all conditions $(C_1), \cdots, (C_m)$ are fulfilled. If $z \in \bigcup_{i=1}^{m} \mathring{L}_i$, then $\infty = \lim_{n\to\infty} Q(\theta_n) \leqq Q(\theta_o)$ by Lemma 1. This is a contradiction. If $z = (u, \cdots, u) \in \mathring{L}_o$, then $\bar{Q}(u) \leqq Q(\theta_o) < \bar{Q}(\bar{u})$, which contradicts $\bar{u} \in U$. Let $z = (z_1, \cdots, z_m) \in \partial_s F(\Theta) \setminus \bigcup_{i=0}^{m} \mathring{L}_i$. Then $z_1 = \cdots = z_m = 0$ or there is k $(1 \leqq k \leqq m)$ such that $z_i = 0$ $(i < k)$ and $z_i = 1$ $(k \leqq i)$. Since there is j such that $0 < y_j < 1$, $\lim_{n\to\infty} Q(\theta_n) = \infty \leqq Q(\theta_o)$. This is a contradiction. Thus K is compact. This completes the proof.

§ 4. Case: $w(u) = [u(1-u)]^{-1}$

In this section we consider the special case: $w(u) = [u(1-u)]^{-1}$. It is easily verified that (1. 1) and (1. 2) are satisfied. In statistical applications this case often arises (see [1] and [2]).

Our main result in this section is the following:

THEOREM 3. *Assume that there exist* j *and* j' *such that* $0 < y_j < y_{j'} < 1$ *and that* $T(x_m) - T(x_1) < 0$. *Then the problem* (P) *has an optimal solution.*

To prove this, we prepare several lemmas.

LEMMA 3 (*cf.* [2; *Proposition* 1]). *Assume that* $0 < y_i < 1$ *for some* i. *Then there exists a unique value* $\bar{u}$ *of* u *which minimizes* $\bar{Q}(u)$ *over* (0, 1). *Moreover* $\bar{u}$ *is a solution of the equation*

(4.1) $$(m - 2\sum_{i=1}^{m} y_i)u^2 + 2(\sum_{i=1}^{m} y_i^2)u - \sum_{i=1}^{m} y_i^2 = 0.$$

LEMMA 4 (*cf.* [2; *Corollary* 1]). *Let* $\{a_i\}$ *and* $\{b_i\}$ *be increasing sequences of* m *real numbers. Then*

(4.2) $$m\sum_{i=1}^{m} a_i b_i \geqq (\sum_{i=1}^{m} a_i)(\sum_{i=1}^{m} b_i).$$

In particular, the strict inequality holds in (4.2) *if* $a_m - a_1 > 0$ *and* $b_m - b_1 > 0$.

LEMMA 5 (*cf.* [2; *Corollary* 2]). *Let* $\{a_i\}$, $\{b_i\}$ *and* $\{c_i\}$ *be increasing sequences such as* $\sum_{i=1}^{m} b_i > 0$. *Assume that* $a_i = a_j$ *if and only if* $b_i = b_j$. *If* $(a_i - a_j)/(b_i - b_j) \geqq (\sum_{i=1}^{m} a_i)/(\sum_{i=1}^{m} b_i)$ *for every* i *and* j *such as* $b_i \neq b_j$, *then*

(4.3) $$(\sum_{i=1}^{m} a_i)(\sum_{i=1}^{m} b_i c_i) \leqq (\sum_{i=1}^{m} a_i c_i)(\sum_{i=1}^{m} b_i).$$

In particular, the strict inequality holds in (4.3) *if* $c_m - c_1 > 0$ *and there exist* i *and* j *such that* $b_i \neq b_j$ *and* $(a_i - a_j)/(b_i - b_j) > (\sum_{i=1}^{m} a_i)/(\sum_{i=1}^{m} b_i)$.

LEMMA 6. *Let* $\bar{u}$ *be a solution of* $d\bar{Q}(u)/du = 0$ *in* (0, 1), *let* $\bar{y}$ *be the mean value of* $\{y_i\}$ *and let* $y_m - y_1 > 0$. *Then*

(i) $\bar{y} < 1/2$ *if and only if* $\bar{y} < \bar{u} < 1/2$.

(ii) $\bar{y} = 1/2$ *if and only if* $\bar{y} = \bar{u}$.

(iii) $\bar{y} > 1/2$ *if and only if* $1/2 < \bar{u} < \bar{y}$.

PROOF. Put $A = \sum_{i=1}^{m}(2y_i - 1)$, $B = \sum_{i=1}^{m} y_i^2$ and $h(u) = -Au^2 + 2Bu - B$. Then $\bar{y} = (A + m)/2m$ and

$$\begin{aligned} h(\bar{y}) &= -A(A+m)^2/4m^2 + B(A+m)/m - B \\ &= A[4mB - (A+m)^2]/4m^2 \\ &= A[mB - (\sum_{i=1}^{m} y_i)^2]/m^2. \end{aligned}$$

On the other hand, $h(1/2) = -A/4$, from which

(4.4) $$h(\bar{y}) = 4h(1/2)[(\sum_{i=1}^{m} y_i)^2 - mB]/m^2.$$

By Lemma 4, $(\sum_{i=1}^{m} y_i)^2 - mB < 0$, so that (4.4) yields (i), (ii) and (iii).

PROOF OF THEOREM 3. Let $\bar{u}$ be a value of u minimizing $\bar{Q}(u)$ over (0, 1), and put $a_i = 2y_i - 1$, $b_i = y_i^2$ and $c_i = -T(x_i)$. Then

$$m\sum_{i=1}^{m} y_i\, T(x_i) - \sum_{i=1}^{m} T(x_i) \sum_{i=1}^{m} y_i = (\sum_{i=1}^{m} a_i \sum_{i=1}^{m} c_i - m \sum_{i=1}^{m} a_i c_i)/2,$$

$$m\sum_{i=1}^{m} y_i^2\, T(x_i) - \sum_{i=1}^{m} T(x_i) \sum_{i=1}^{m} y_i^2 = \sum_{i=1}^{m} b_i \sum_{i=1}^{m} c_i - m \sum_{i=1}^{m} b_i c_i,$$

$$\sum_{i=1}^{m} y_i \sum_{i=1}^{m} y_i^2\, T(x_i) - \sum_{i=1}^{m} y_i^2 \sum_{i=1}^{m} y_i\, T(x_i) = (\sum_{i=1}^{m} b_i \sum_{i=1}^{m} c_i - m \sum_{i=1}^{m} b_i c_i)/2$$

$$+ (\sum_{i=1}^{m} b_i \sum_{i=1}^{m} a_i c_i - \sum_{i=1}^{m} a_i \sum_{i=1}^{m} b_i c_i)/2,$$

so that

$$2G(u) = (\sum_{i=1}^{m} c_i)\,[(\sum_{i=1}^{m} a_i)u^2 - 2(\sum_{i=1}^{m} b_i)\,u + \sum_{i=1}^{m} b_i]$$

$$-m(\sum_{i=1}^{m} a_i c_i)u^2 + 2m(\sum_{i=1}^{m} b_i c_i)u - m\sum_{i=1}^{m} b_i c_i + K,$$

where $K = \sum_{i=1}^{m} b_i \sum_{i=1}^{m} a_i c_i - \sum_{i=1}^{m} a_i \sum_{i=1}^{m} b_i c_i$. By (4. 1)

$$(4.5)\quad 2G(\bar{u}) = m[-(\sum_{i=1}^{m} a_i c_i)\bar{u}^2 + 2(\sum_{i=1}^{m} b_i c_i)\bar{u} - \sum_{i=1}^{m} b_i c_i] + K.$$

Put $\overline{G} = -(\sum_{i=1}^{m} a_i c_i)\bar{u}^2 + 2(\sum_{i=1}^{m} b_i c_i)\bar{u} - \sum_{i=1}^{m} b_i c_i$. It then follows from (4. 1) that $(\sum_{i=1}^{m} a_i)\overline{G} = K(1 - 2\bar{u})$. Since $2G(\bar{u}) = m\overline{G} + K$,

$$(4.6)\quad (\sum_{i=1}^{m} a_i)G(\bar{u}) = mK(\bar{y} - \bar{u}).$$

In order to show that $K > 0$ we have only to verify that sequences $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ satisfy conditions of Lemma 5. It is clear by our assumption that sequences $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ are increasing, $b = \sum_{i=1}^{m} b_i > 0$ and $c_m - c_1 > 0$, and $a_i = a_j$ if and only if $b_i = b_j$. In case that $b_i \neq b_j$, we put $A = (a_i - a_j)/(b_i - b_j) = 2/(y_i + y_j)\,(>0)$ and $B = (\sum_{i=1}^{m} a_i)/(\sum_{i=1}^{m} b_i) = m(2\bar{y} - 1)/b$. It is clear that $B < A$ if $\bar{y} \leqq 1/2$. If $\bar{y} > 1/2$, it can be shown that $\bar{y}^2/(2\bar{y} - 1)$ and $\bar{y}^2 \leqq b/m$ by Schwartz's inequality, so that $\bar{y}^2/(2\bar{y} - 1) \leqq b/[m(2\bar{y} - 1)]$. Therefore $m(2\bar{y} - 1)/b < 1$. Since $y_i \neq y_j$, $y_i + y_j < 2$, so that $B < A$. Thus $K > 0$. Next we shall prove that $G(\bar{u}) > 0$. Since $K > 0$, by the aid of Lemma 6 and (4. 6) we conclude that $G(\bar{u}) > 0$ if $\bar{y} \neq 1/2$. In case that $\bar{y} = 1/2$, we have by Lemma 6 and (4. 5)

$$8G(1/2) = (4b - m)\sum_{i=1}^{m} a_i c_i.$$

It follows from Lemma 4 that $\sum_{i=1}^{m} a_i c_i > 0$ and $(\sum_{i=1}^{m} y_i)^2 < mb$. Since $\sum_{i=1}^{m} y_i = m/2$, $m/4 < b$ and hence $4b - m > 0$. Thus $G(1/2) > 0$. Since $G(\bar{u}) > 0$, our assertion follows from Theorem 2.

References

[1] T. W. Anderson and D. A. Darling; A test of goodness of fit, J. Amer. Stat. Assn. **49** (1953), 765-769.

[2] T. Nakamura and T. Kariya; On the weighted least squares estimation and the existence theorem of its optimal solution, Kawasaki Medical Liberal Arts & Science Course **1** (1975), 1-11.