# A Permutation S-sample Test Against Ordered Alternatives Based on Interval Data Samples

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*Accepted for publication on September 7, 1995*

**ABSTRACT.** A distribution-free test is proposed that is an extension of the test by Jonckheere to interval data samples. Interval data are often obtained in the experiments in which the observation for each subject is specified only by an interval. We define a generalized sign of difference between two observations based on their interval data under the estimated distribution of each observation. The test statistic J is based on these generalized signs instead of ranks.

Using the statistic J based on generalized signs, the hypothesis of no difference among the s treatments is tested against the alternatives of a definite order of these treatments. When s and sample sizes are small, we can derive the probability distribution of test statistic J exactly, but for large values of s and sample sizes the computations are impracticable. However, as J is then approximately normally distributed about zero, one requires the variance of J. We illustrate an easy calculation method of the variance of J and present a numerical example.

**Key words : permutation test — ordered alternatives — interval data — generalized sign**

We sometimes meet with the problem of testing the hypothesis of equality of s treatments against the alternatives of a definite order of these treatments, where the observation for each individual is specified only by an interval.

For example, in testing the effectiveness of class size on learning, one may wish to test the hypothesis of no effect against the alternatives that the effectiveness increases as the size of the class decreases successively from more than 100 students, 100-60, 60-30, and fewer than 30. In this case, the treatments are ordered (of course, before the responses have been observed) in such a way that under the alternatives to null hypothesis H one would expect larger responses under treatment 2 than under treatment 1, under treatment 3 than under treatment 2, and so on.

Similarly, when one compares groups with different degrees of stress to evaluate the performance of some task of manual dexterity, we have an analogous problem. In such cases, the Kruskal-Wallis test is no longer appropriate.

The test by Jonckheere (1954)[1,2] is suitable against such ordered alternatives, and a table of the null distribution of J for equal group sizes $n_1=n_2=\cdots=n_s=n$ for several combinations of small values of s and n is given. But only the probabilities up to 0.5 are tabled.

For large values of $N=n_1+n_2+\cdots+n_s$, the statistic J is approximately

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normally distributed, and then one requires the expectation and variance of J to apply this approximation. When data are ordinary, it is known that E(J) = 0 and $V(J) = \{N^2 \cdot (2n_i+3) - \sum_{i=1}^{s} n_i^2 \cdot (2n_i+3)\}/18$, but in the case of interval data we cannot use the formula for V(J). Therefore, an easy calculation method of V(J) is necessary.

Now, a continuous random variable $X$ is said to be interval-censored into a non-zero interval I if the only information about a realized value of $X$ is that it lies in I. The interval-censoring of a realized value is a very common procedure in biometrics, and interval-censored data are often obtained in multistage follow-up examination. However, the situations in which specifying a realized value by an interval is appropriate are not limited to interval-censoring.

In the case of some characteristics of a living individual, the values may change greatly from moment to moment as a result not only of physical causes but also mental ones. This paper extends Jonckheere's procedure for ordinary observations to the interval data samples.[3,4,5]

## The test proposed

Let $(X_{11}, X_{12}, \ldots, X_{1n_1}), \ldots, (X_{i1}, X_{i2}, \ldots, X_{in_i}), \ldots, (X_{s1}, X_{s2}, \ldots, X_{sn_s})$ be samples of size $n_1, \ldots, n_i, \ldots, n_s$, randomly drawn from populations with continuous cumulative distributions $F_1(X), \ldots, F_i(X), \ldots, F_s(X)$, respectively, and arranged in such a manner that the first suffix of the $X$'s is in the order implied by the alternative hypothesis $F_1(X) > F_2(X) > \cdots > F_i(X) > \cdots > F_s(X)$ for all $X$.

And let $X_{i\alpha_i}$ be specified by an interval datum $(x_{i\alpha_i L}, x_{i\alpha_i U})$; that is to say, let the realized value of $X_{i\alpha_i}$ be an element of $(x_{i\alpha_i L}, x_{i\alpha_i U})$.

Generally, if $X_{i\alpha_i}$ is the $\alpha_i$th value in the $i$th sample drawn from a population with c. d. f. $F_i(X)$, we wish to test the null hypothesis H that $F_i(X) = F_j(X)$ $(i, j=1, 2, \ldots, s; i \neq j)$, against the alternative A that $F_i(X) > F_j(X)$ $(i<j)$ for all $X$.

We define a generalized sign[3,4] of $X_{i\alpha_i} - X_{j\alpha_j}$ to be $d_{i\alpha_i j\alpha_j}$ based on their interval data $(x_{i\alpha_i L}, x_{i\alpha_i U})$ and $(x_{j\alpha_j L}, x_{j\alpha_j U})$, in the following manner:

$$\begin{aligned} d_{i\alpha_i j\alpha_j} &= E(sgn(X_{i\alpha_i} - X_{j\alpha_j})) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} sgn(x_i - x_j) f_i(x_i; x_{i\alpha_i L}, x_{i\alpha_i U})\, g_j(x_j; x_{j\alpha_j L}, x_{j\alpha_j U})\, dx_i dx_j, \qquad \cdots\cdots\cdots (1) \end{aligned}$$

where $f_i(x_i; x_{i\alpha_i L}, x_{i\alpha_i U})$ is some probability density function which shows the variation of measured values for the $\alpha_i$th individual in the $i$th sample, independent of $F_i(X)$, and $g_j(x_j; x_{j\alpha_j L}, x_{j\alpha_j U})$ is an analogous p. d. f. for the $\alpha_j$th individual in the $j$th sample, independent of $F_j(X)$, and

$$sgn(X_i - X_j) = \begin{cases} 1 & \text{if } X_i > X_j, \\ 0 & \text{if } X_i = X_j, \\ -1 & \text{if } X_i < X_j. \end{cases}$$

Then it is easy to see that $d_{i\alpha_i i\alpha_i} = 0$ and $d_{i\alpha_i j\alpha_j} = -d_{j\alpha_j i\alpha_i} (i \neq j)$.

Let $J_{ij} = \sum_{\alpha_i=1}^{ni} \sum_{\alpha_j=1}^{nj} d_{i\alpha_i j\alpha_j} (i<j)$, and $J = \sum_{i=1}^{s-1} \sum_{j=1+i}^{s} J_{ij}$, $\qquad \cdots\cdots\cdots\cdots\cdots (2)$

It is the statistic J that we propose to use for testing the null hypothesis H against the alternative of ordered cumulative distribution functions when data are specified by intervals. This statistic J corresponds to Jonckheere's statistic S, and these statistical procedures are also applicable to the case of ordinary data; that is to say, if a realized value of $X_{ia_i}$ is specified by $x_{ia_i}$ ordinarily, we have only to make $x_{ia_iL}=x_{ia_iU}=x_{ia_i}$.

To make it short and clear, we assume that the observed values: $x_{ia_iL}$, $x_{ia_iU}$ are all integers obtained by rounding of measured values. If necessary, we have only to multiply the measured values by $10^m$.

And in the practical integration, we assume that

$$f_i(x_i\,;\,x_{ia_iL},\,x_{ia_iU})=0 \quad \text{if } x_i<x_{ia_iL}-.5 \text{ or } x_i>x_{ia_iU}+.5, \text{ and}$$

$$\int_{x_{ia_iL}-.5}^{x_{ia_iU}+.5} f_i(x_i\,;\,x_{ia_iL},\,x_{ia_iU})\,dx_i=1,$$

and that

$$g_j(x_j\,;\,x_{ja_jL},\,x_{ja_jU})=0 \quad \text{if } x_j<x_{ja_jL}-.5 \text{ or } x_j>x_{ja_jU}+.5, \text{ and}$$

$$\int_{x_{ja_jL}-.5}^{x_{ja_jU}+.5} g_j(x_j\,;\,x_{ja_jL},\,x_{ja_jU})\,dx_j=1.$$

Using (1) we can then get

$$d_{ia_ija_j}=\begin{cases} -1 \cdots\cdots \text{if } x_{ia_iU}<x_{ja_jL}, \\ +1 \cdots\cdots \text{if } x_{ja_jU}<x_{ia_iL}, \\ \text{a real number which is larger than } -1 \\ \qquad\qquad \text{and less than } +1\cdots\cdots\text{otherwise.} \end{cases}$$

It is a simple task to obtain the conditional exact distribution of J under the null hypothesis H when $N=\sum_{i=1}^{s} n_i$ is not large. Since the N individuals can be labeled differently, there are $\binom{N}{n_1, n_2, \cdots, n_s}$ possible assignments of the subjects to s samples with $n_1, n_2, \ldots, n_s$ observations, where $\binom{N}{n_1, n_2, \cdots, n_s}=\mathrm{N}!/\prod_{i=1}^{s} n_i!$. This is independent of whether there are tied observations in the N combined observations or not. Under the null hypothesis H, each of these $\binom{\mathrm{N}}{n_1, n_2, \cdots, n_s}$ possible assignments occurs with the same probability and so we can construct the conditional exact distribution of J by all the values of J which are calculated for these assignments, respectively. Then the significance probability (also called the P-value) of the observed s samples is as follows:

P-value=(the number of assignments in which the values of J are not larger than $\mathrm{J}_{(1)})\cdot\prod_{i=1}^{s} n_i!/\mathrm{N}!$, where $\mathrm{J}_{(1)}$ is the value for the observed samples.

By the P-value, we can decide to reject or accept the null hypothesis H against the alternative A.

**The conditional mean and variance of J**

Unfortunately, the computations required make it impracticable to carry out such a permutation test exactly (except for small N). For most practical purposes, however, the approximately normal distribution of J can be derived by a similar form of argument with the Jonckheere's S.[1)]

Now, let the conditional mean and variance of J be denoted by $\mathrm{E}(\mathrm{J}|\mathrm{P},\mathrm{H})$ and $\mathrm{V}(\mathrm{J}|\mathrm{P},\mathrm{H})$, respectively, where P is the pattern of observed interval data. The expectations are obtained by summing over all the $\mathrm{N}!/\prod_{i=1}^{s} n_i!$ equally likely samples leading to the same observed pattern P.

It is easy to see that

$$\mathrm{E}(\mathrm{J}|\mathrm{P},\mathrm{H})=0 \qquad \cdots\cdots\cdots\cdots\cdots (3)$$

by the randomness of data allocation to s samples.
Then the variance of J is given as follows:

$$\mathrm{V}(\mathrm{J}|\mathrm{P},\mathrm{H})=\mathrm{E}(\mathrm{J}^2|\mathrm{P},\mathrm{H})=\sum_{c=1}^{w}\mathrm{J}_{(c)}^2/w, \qquad \cdots\cdots\cdots\cdots\cdots (4)$$

where $\mathrm{J}_{(c)}$ is the value of J for the $c$th data allocation, and $w=\mathrm{N}!/\prod_{i=1}^{s} n_i!$.

In order to find an easy calculation method for $\mathrm{V}(\mathrm{J}|\mathrm{P},\mathrm{H})$, we number the data: $(x_{11}, x_{12}, \ldots, x_{1n_1})$, $(x_{21}, x_{22}, \ldots, x_{2n_2}), \ldots, (x_{s1}, x_{s2}, \ldots, x_{sn_s})$ serially from 1 to N, and indicate them by $x_{(1)}, x_{(2)}, \ldots, x_{(N)}$, respectively. We express the generalized sign of $X_{(i)}-X_{(j)}$ by $\mathrm{D}_{ij}$, and let $\mathrm{D}_{i\cdot}=\sum_{j=1}^{N}\mathrm{D}_{ij}$. Then it is clear that $\mathrm{D}_{ii}=0$ $(i=1,\ldots,\mathrm{N})$, and $\mathrm{D}_{ij}=-\mathrm{D}_{ji}(i,j=1,\ldots,\mathrm{N}\,;i\neq j)$.

Let $\mathrm{J}_{ij}=\sum_{i=n_1+\cdots+n_{i-1}+1}^{n_1+n_2+\cdots+n_i}\ \sum_{j=n_1+\cdots+n_{j-1}+1}^{n_1+n_2+\cdots+n_j}\mathrm{D}_{ij}$, and $\mathrm{J}=\sum_{i=1}^{s-1}\sum_{j=1+i}^{s}\mathrm{J}_{ij}$. $\qquad \cdots\cdots\cdots\cdots (5)$

(5) corresponds to (2).

It is easy to see that
$\{\mathrm{D}_{ij}\}(i,j=1,\ldots,\mathrm{N})$ is equal to $\{d_{i\alpha_i j\alpha_j}\}(i,j=1,\ldots,s\,;\alpha_i=1,\ldots,n_i,\alpha_j=1,\ldots,n_j)$,
the number of kinds of $\mathrm{D}_{ij}(i\neq j)$ is $\mathrm{N}(\mathrm{N}-1)$, and
each $\mathrm{J}_{(c)}(c=1,\ldots,w)$ contains $t$ terms of $\mathrm{D}_{ij}(i\neq j)$, where $t=\sum_{i=1}^{s-1}\sum_{j=1+i}^{s}n_i\cdot n_j$.

$$\text{Now, } \sum_{i=1}^{N}\mathrm{D}_{i\cdot}^2=\sum_{i=1}^{N}(\sum_{j=1}^{N}\mathrm{D}_{ij})^2=\sum_{i=1}^{N}(\mathrm{D}_{i1}^2+\cdots+\mathrm{D}_{ii-1}^2+\mathrm{D}_{ii+1}^2+\cdots+\mathrm{D}_{iN}^2)$$
$$+\underset{(i\neq j,\ i\neq k,\ j\neq k)}{\sum_{i=1}^{N}\ \sum_{j=1}^{N}\ \sum_{k=1}^{N}}\mathrm{D}_{ij}\cdot\mathrm{D}_{ik}. \qquad \cdots\cdots\cdots\cdots (6)$$

In the expansion of $\sum_{i=1}^{N}\mathrm{D}_{i\cdot}^2$, each $\mathrm{D}_{ij}(i,j=1,\ldots,\mathrm{N}\,;i\neq j)$ and each $\mathrm{D}_{ij}\cdot\mathrm{D}_{ik}$ $(i,j,k=1,\ldots,\mathrm{N}\,;i\neq j,\ i\neq k,j\neq k)$ appear only once, respectively.

On the other hand, $\mathrm{D}_{ij}\cdot\mathrm{D}_{kl}$'s$(i\neq j,\ k\neq l)$ in the expansion of $\sum_{c=1}^{w}\mathrm{J}_{(c)}^2$ are classified into three groups. These groups correspond to cases where ( i ) $i=k$ and $j=l$, ( ii ) two of $i,j,k$, and $l$ are equal, and (iii) $i,j,k$, and $l$ all differ from each other.

Let $\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{kl})$ be the frequency of occurrence of $\mathrm{D}_{ij}\cdot\mathrm{D}_{kl}$ in the expansion of $\sum_{c=1}^{w}\mathrm{J}_{(c)}^2$.

It is easy to see that
$\mathrm{D}_{ij}\cdot\mathrm{D}_{kl}$'s belonging to the third group vanish by summation,
$\#(\mathrm{D}_{ij}^2)$ $(i\neq j)$ in the first group is equal to $t\cdot w/\mathrm{N}(\mathrm{N}-1)$ for all combinations of $i$ and $j$ $(i,j=1,\ldots,\mathrm{N}\,;i\neq j)$,
and in the second group, $\mathrm{D}_{ij}\cdot\mathrm{D}_{ik}=\mathrm{D}_{ji}\cdot\mathrm{D}_{ki}=-\mathrm{D}_{ij}\cdot\mathrm{D}_{ki}=-\mathrm{D}_{ji}\cdot\mathrm{D}_{ik}$,
$\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{ik})=\#(\mathrm{D}_{ji}\cdot\mathrm{D}_{ki})$, and $\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{ki})=\#(\mathrm{D}_{ji}\cdot\mathrm{D}_{ik})$, where $i\neq j$, $i\neq k$, and $j\neq k$. Accordingly, the sum of these terms is equal to $\{\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{ik})+\#(\mathrm{D}_{ji}\cdot\mathrm{D}_{ki})-\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{ki})-\#(\mathrm{D}_{ji}\cdot\mathrm{D}_{ik})\}\cdot(\mathrm{D}_{ij}\cdot\mathrm{D}_{ik})$.
Similarly, for all combinations of $i,j$, and $k\,(i,j,k=1,\ldots,\mathrm{N}\,;i\neq j,\ i\neq k,\ j\neq k)$, $\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{ik})+\#(\mathrm{D}_{ji}\cdot\mathrm{D}_{ki})-\#(\mathrm{D}_{ij}\cdot\mathrm{D}_{ki})-\#(\mathrm{D}_{ji}\cdot\mathrm{D}_{ik})$ has the same value in

the expansion of $\sum_{c=1}^{w} J^2_{(c)}$.

Consequently, we can obtain the value of $V(J|P, H)$ using the values of $\sum_{i=1}^{N}\sum_{j=1}^{N} D^2_{ij}\,(i \neq j)$ and $\underset{(i\neq j,\ i\neq k,\ j\neq k)}{\sum_{i=1}^{N}\ \sum_{j=1}^{N}\ \sum_{k=1}^{N}} D_{ij}\cdot D_{ik}$ in (6).

We can derive $\#(D_{ij}\cdot D_{ik})$ under any sample sizes $n_1, n_2, \ldots, n_s$, but the driving procedure is not simple. Therefore, we will show the principle of the deriving procedure in the case of $n_1=n_2=n_3=3$, and $N=9$. According to the randomness of data allocation to s samples under H, we only have to derive the frequency of occurrence of $D_{12}\cdot D_{13}$, $D_{21}\cdot D_{31}$, $D_{12}\cdot D_{31}$, and $D_{21}\cdot D_{13}$.

It is clear that $s=3$, $t=27$, $w=1680$, $N(N-1)=72$, and $\#(D^2_{12})=t\cdot w/N(N-1)=630$.

Since $D_{12}\cdot D_{13}$ occurs when the data $x_{(1)}$, $x_{(2)}$, and $x_{(3)}$ are allocated to ordered three samples as $(x_{(1)},\cdot,\cdot)\ (x_{(2)}, x_{(3)},\cdot)\ (\cdot,\cdot,\cdot)$, $(x_{(1)},\cdot,\cdot)\ (\cdot,\cdot,\cdot)\ (x_{(2)}, x_{(3)},\cdot)$, $(\cdot,\cdot,\cdot)\ (x_{(1)},\cdot,\cdot)\ (x_{(2)}, x_{(3)},\cdot)$, $(x_{(1)},\cdot,\cdot)\ (x_{(2)},\cdot,\cdot)\ (x_{(3)},\cdot,\cdot)$ or $(x_{(1)},\cdot,\cdot)\ (x_{(3)},\cdot,\cdot)\ (x_{(2)},\cdot,\cdot)$,

$\#(D_{12}\cdot D_{13}) = \binom{N-3}{2}\binom{N-5}{1}\binom{s}{2} + \binom{N-3}{2}\binom{N-5}{2}\cdot 2$
$= \binom{6}{2}\binom{4}{1}\binom{3}{2} + \binom{6}{2}\binom{4}{2}\cdot 2 = 360.$

Because $D_{12}\cdot D_{31}$ occurs when the data $x_{(1)}$, $x_{(2)}$, and $x_{(3)}$ are allocated to ordered three samples as $(x_{(3)},\cdot,\cdot)\ (x_{(1)},\cdot,\cdot)\ (x_{(2)},\cdot,\cdot)$, $\#(D_{12}\cdot D_{31}) = \binom{N-3}{2}\binom{N-5}{2} = \binom{6}{2}\binom{4}{2} = 90$.

By the randomness of data allocation to three samples, we have $\#(D_{21}\cdot D_{31}) = \#(D_{12}\cdot D_{13})$ and $\#(D_{21}\cdot D_{13}) = \#(D_{12}\cdot D_{31})$.
Therefore, $\#(D_{12}\cdot D_{13}) + \#(D_{21}\cdot D_{31}) - \#(D_{12}\cdot D_{31}) - \#(D_{21}\cdot D_{13}) = 540$.

To make it short and clear, we assume that data are ordinary (not interval) and they are all different (not tied). Then, in (6),
$D_{ij}\,(i\neq j)=1$ or $-1$, and so $\sum_{i=1}^{N}\sum_{j=1}^{N} D^2_{ij} = N(N-1)=72$. And $\sum_{i=1}^{N} D^2_{i\cdot} = (N-1)^2 + (N-2)^2 + \cdots + (1-N)^2 = (-8)^2+(-6)^2+(-4)^2+(-2)^2+0^2+2^2+4^2+6^2+8^2 = 240$.

$$\therefore \underset{(i\neq j,\ i\neq k,\ j\neq k)}{\sum_{i=1}^{N}\ \sum_{j=1}^{N}\ \sum_{k=1}^{N}} D_{ij}\cdot D_{ik} = 240-72=168.$$

Consequently, $V(J|P, H) = (630\cdot 72 + 540\cdot 168)/1680 = 81$. This agrees quite well with the exact value.

**A numerical example**

Suppose that the following interval data are obtained in an experiment involving four independent groups :

I (18, 21), (20, 24), (36, 40), (53, 59)
II (22, 26), (36, 41), (42, 45), (48, 54), (50, 56)
III (26, 31), (42, 47), (46, 52), (68, 75)
IV (29, 33), (60, 63), (80, 85)

Each member took the test several times and the observed values were summarized in their range as an interval datum.

The experimenter wishes to test the hypothesis that the four samples have come from the same population against the alternative that the populations are such that the values from samples Ⅰ, Ⅱ, Ⅲ, Ⅳ are in an expected order of increasing value.

We assume that observing values of each member are uniformly distributed in the interval. For the computation of J we have

$n_1=4$, $n_2=5$, $n_3=4$, $n_4=3$, $s=4$, $N=16$, $t=95$, $w=50450400$.

$D_{12}(=d_{11,12})=-.8$, $D_{13}(=d_{11,13})=-1, \ldots, D_{25}(=d_{12,21})=-.64, \ldots,$

$D_{52}(=d_{21,12})=+.64, \ldots, D_{16,15}(=d_{43,42})=+1,$

$J_{12}=-7.2148$, $J_{13}=-8.0000$, $J_{14}=-8.0000$, $J_{23}=-3.9939$, $J_{24}=-7.0000$,

$J_{34}=-3.7000$.

Hence, using (5), $J_{(1)}=-37.9087$.

And for the computation of $V(J|P, H)$ we have

$$D_{1\cdot}^2+D_{2\cdot}^2+\ldots,+D_{16\cdot}^2=(-14.8000)^2+(-12.8400)^2+.2861^2+\cdots+15.0000^2$$
$$=1314.0860,$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}D_{ij}^2=227.7490, \quad \underset{(i\neq j,\ i\neq k,\ j\neq k)}{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}} D_{ij}\cdot D_{ik}=\sum_{i=1}^{N}D_{i\cdot}^2-\sum_{i=1}^{N}\sum_{j=1}^{N}D_{ij}^2=1117.9500.$$

In the same manner as in the previous paragraph, we have

$$\#(D_{12}\cdot D_{13})=\#(D_{21}\cdot D_{31})=\binom{N-3}{2}\cdot\binom{N-6}{3}\binom{N-9}{4}+\binom{N-3}{3}\binom{N-6}{5}\binom{N-11}{2}+\binom{N-3}{3}\binom{N-6}{5}\binom{N-11}{4}+\binom{N-3}{4}\binom{N-7}{4}\binom{N-11}{2}+\binom{N-3}{4}\binom{N-7}{4}\binom{N-11}{4}+\binom{N-3}{4}\binom{N-7}{5}\binom{N-12}{3}+\left\{\binom{N-3}{3}\binom{N-6}{4}\binom{N-10}{3}+\binom{N-3}{3}\cdot\binom{N-6}{4}\binom{N-10}{4}+\binom{N-3}{3}\binom{N-6}{5}\binom{N-11}{3}+\binom{N-3}{4}\binom{N-7}{5}\binom{N-12}{3}\right\}\cdot 2=10360350,$$

$$\#(D_{21}\cdot D_{13})=\#(D_{12}\cdot D_{31})=\binom{N-3}{3}\binom{N-6}{4}\binom{N-10}{3}+\binom{N-3}{3}\binom{N-6}{4}\binom{N-10}{4}+\binom{N-3}{3}\binom{N-6}{5}\binom{N-11}{3}+\binom{N-3}{4}\binom{N-7}{4}\binom{N-11}{3}=3183180.$$

Accordingly, $\#(D_{12}\cdot D_{13})+\#(D_{21}\cdot D_{31})-\#(D_{21}\cdot D_{13})-\#(D_{12}\cdot D_{31})=14354340$.

Now, $\#(D_{12}^2)=t\cdot w/N(N-1)=95\cdot 50450400/240=19969950$, and we have

$$V(J|P, H)=(19969950\cdot 227.7490+14354340\cdot 1117.9500)/50450400$$
$$=408.2330=20.2048^2.$$

Therefore, the statistic J is approximately normally distributed with mean 0 and variance $20.2048^2$.

As $J_{(1)}$ (the value of J for the observed samples) is $-37.9087$, the P-value is 0.0303. Thus the experimenter could with some confidence reject the null hypothesis, and accept an alternative that the sample came from populations which were stochastically ordered in the series Ⅰ, Ⅱ, Ⅲ, and Ⅳ.

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